14.6. Directional derivatives and the gradient vector

Def Let $f(x,y,z)$ be a differentiable function.

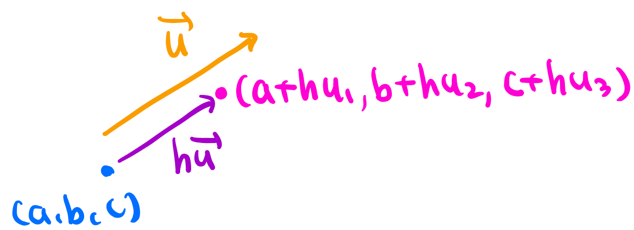
(1) Its gradient is given by

$$\nabla f := (f_x, f_y, f_z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

(2) Its directional derivative at (a,b,c) along a unit vector $\vec{u} = (u_1, u_2, u_3)$ is defined by

$$D_{\vec{u}} f(a,b,c) := \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$$

"change rate in the direction of $\vec{u}$".



Note (1) The gradient is a vector, while the directional derivatives are scalars.

(2) The partial derivatives are special cases of the directional derivatives:

$$D_{\vec{i}} f = \frac{\partial f}{\partial x}, \quad D_{\vec{j}} f = \frac{\partial f}{\partial y}, \quad D_{\vec{k}} f = \frac{\partial f}{\partial z}$$

where $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$.

Prop Let $f(x,y,z)$ be a differentiable function.

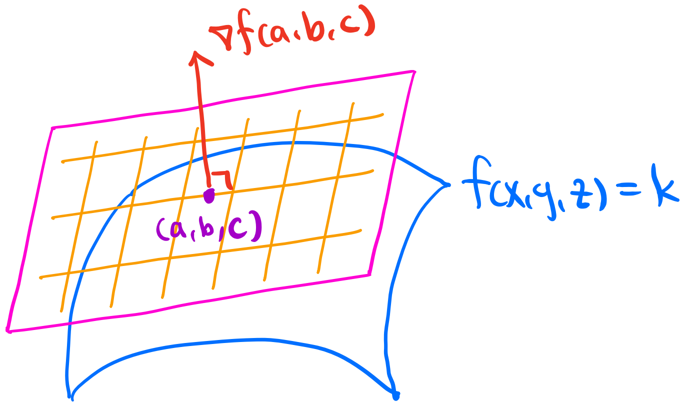
(1) $D_{\vec{u}} f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$

(2) $\nabla f(a,b,c)$ points in the direction of fastest increase at (a,b,c), whereas $-\nabla f(a,b,c)$ points in the direction of fastest decrease at (a,b,c).

(3) $|\nabla f(a,b,c)|$ is equal to the maximum directional derivative at (a,b,c), while $-|\nabla f(a,b,c)|$ is equal to the minimum directional derivative at (a,b,c).

most negative

(4) $\nabla f(a,b,c)$ is a normal vector of the tangent plane to the level surface of f at (a,b,c).



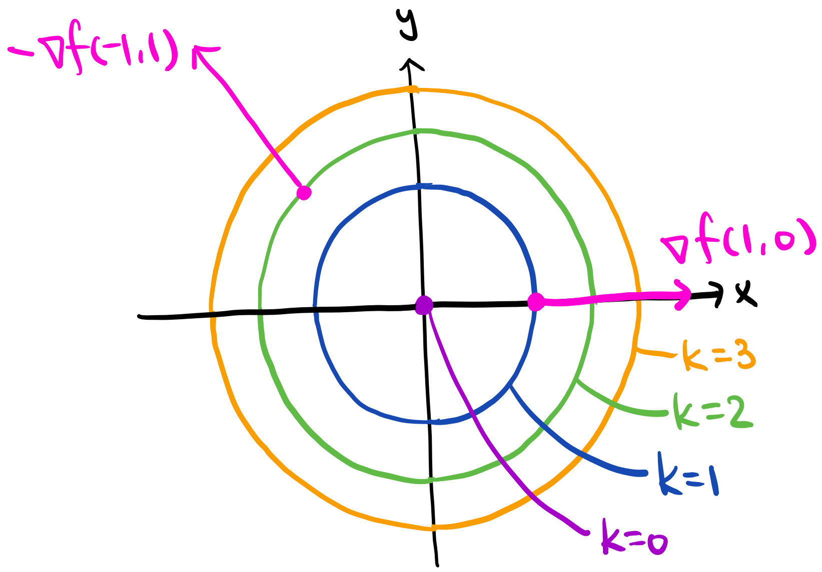
(5) If x, y, z are given by a vector function $\vec{r}(t)$, then $\dfrac{df}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

Ex Consider $f(x,y) = x^2+y^2$.

(1) Draw a contour map of $f(x,y)$ with levels at $0, 1, 2, 3$.

Sol Level curve at k is given by $x^2+y^2=k$

$\rightsquigarrow$ a circle of radius $\sqrt{k}$ and center $(0,0)$.



(2) Draw $\nabla f(1,0)$ and $\nabla f(-1,1)$ on the contour map.

Sol $\nabla f = (f_x, f_y) = (2x, 2y)$

$\nabla f(1,0) = (2,0)$, $\nabla f(-1,1) = (-2,2)$.

Note Our sketch shows that these vectors point in the direction of fastest increase at each point and are perpendicular to the tangent lines to each level curve.

Ex Consider $g(x,y,z) = x^2+y^2+z^2+xyz$.

(1) Find the directional derivative at $P=(0,1,1)$ along the direction of $\vec{v}=(4,0,3)$

Sol The unit vector of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{(4,0,3)}{\sqrt{4^2+0^2+3^2}} = \frac{1}{5}(4,0,3)$$

$$\nabla g = (g_x, g_y, g_z) = (2x+yz,\ 2y+xz,\ 2z+xy)$$

$$\rightsquigarrow \nabla g(0,1,1) = (1,2,2)$$

$$\Rightarrow D_{\vec{u}}\, g(0,1,1) = \nabla g(0,1,1)\cdot\vec{u}$$

$$= (1,2,2)\cdot\frac{1}{5}(4,0,3) = \boxed{2}$$

(2) Find the unit vector that points in the direction of fastest increase at P.

Sol This is given by the unit vector of $\nabla g(0,1,1)$.

$$\frac{\nabla g(0,1,1)}{|\nabla g(0,1,1)|} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \boxed{\frac{1}{3}(1,2,2)}$$

Note The direction of fastest decrease at P is given by $-\frac{1}{3}(1,2,2)$.

(3) Find the directional derivative at P along the unit vector in (2).

<u>Sol</u> This is equal to the maximum directional derivative at P, given by $|\nabla g(0,1,1)| = \boxed{3}$

(4) Find a unit vector $\vec{w}$ along which the directional derivative at P is zero.

<u>Sol</u> Set $\vec{w} = (a,b,c)$.

$$\Rightarrow D_{\vec{w}}\, g(0,1,1) = \nabla g(0,1,1)\cdot\vec{w}$$
$$= (1,2,2)\cdot(a,b,c)$$
$$= a+2b+2c = 0$$

We can take $\vec{w}$ to be any unit vector with $a+2b+2c=0$.

$$\vec{w} = \frac{(0,1,-1)}{|(0,1,-1)|} = \frac{(0,1,-1)}{\sqrt{0^2+1^2+(-1)^2}} = \boxed{\frac{1}{\sqrt{2}}(0,1,-1)}$$

<u>Note</u> There are infinitely many possible answers.

e.g. $\vec{w} = \frac{1}{\sqrt{5}}(2,-1,0)$ or $\vec{w} = \frac{1}{3}(2,-2,1)$.

Ex For each surface, find an equation of the tangent plane at $(3,2,0)$.

(1) The surface $z = \ln(x-y)$.

Sol $z = \ln(x-y) \Rightarrow \ln(x-y) - z = 0$

$\rightsquigarrow$ a level surface of $f(x,y,z) = \ln(x-y) - z$.

$\nabla f = \left(\frac{1}{x-y}, -\frac{1}{x-y}, -1 \right)$

A normal vector is $\nabla f(3,2,0) = (1,-1,-1)$

The tangent plane is given by

$$\boxed{1\cdot(x-3) - 1\cdot(y-2) - 1\cdot(z-0) = 0}$$

(2) The surface $x^2 - y^3 + 2yz = e^z$

Sol $x^2 - y^3 + 2yz = e^z \Rightarrow x^2 - y^3 + 2yz - e^z = 0$

$\rightsquigarrow$ a level surface of $g(x,y,z) = x^2 - y^3 + 2yz - e^z$

$\nabla g = (2x, -3y^2 + 2z, 2y - e^z)$

A normal vector is $\nabla g(3,2,0) = (6,-12,3)$

The tangent plane is given by

$$\boxed{6(x-3) - 12(y-2) + 3(z-0) = 0}$$

Note The tangent plane formula for graphs can be used for (1), but not for (2).

Ex Suppose that the electric potential at a point is given by $V(x,y,z) = 5x^2 - 3xy + xyz$.

A particle is moving so that its position at time t is given by $\vec{r}(t) = (t^2-1,\ 2-t,\ t)$

Find the change rate of the electric potential along the path of particle at $t=1$.

Sol $\left.\dfrac{dV}{dt}\right|_{t=1} = \nabla V(\vec{r}(1)) \cdot \vec{r}\,'(1)$.

$\nabla V = (V_x, V_y, V_z) = (10x - 3y + yz,\ -3x + xz,\ yz)$

$\vec{r}(1) = (0, 1, 1)$

$\nabla V(\vec{r}(1)) = \nabla V(0,1,1) = (-2, 0, 1)$

$\vec{r}\,'(t) = (2t, -1, 1) \rightsquigarrow \vec{r}\,'(1) = (2, -1, 1)$

$\Rightarrow \left.\dfrac{dV}{dt}\right|_{t=1} = (-2, 0, 1) \cdot (2, -1, 1) = \boxed{-3}$